NONISOTHERMAL COUETTE FLOW OF A NON-NEWTONIAN FLUID UNDER THE ACTION OF A PRESSURE GRADIENT

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Nonisothermal Couette flow has been studied in a number of papers [1-11] for various laws of the temperature dependence of viscosity. In [1] the viscosity of the medium was assumed constant; in [2-5] a hyperbolic law of variation of viscosity with temperature was used; in [6-8] the Reynolds relation was assumed; in [9] the investigation was performed for an arbitrary temperature dependence of viscosity. Flows of media with an exponential temperature dependence of viscosity are characterized by large temperature gradients in the flow. This permits the treatment of the temperature variation in the flow of the fluid as a hydrodynamic thermal explosion [8, 10, 11]. The conditions of the formulation of the problem of the articles mentioned were limited by the possibility of obtaining an analytic solution. In the present article we consider nonisothermal Couette flows of a non-Newtonian fluid under the action of a pressure gradient along the plates. The equations for this case do not have an analytic solution. Methods developed in [12-14] for the qualitative study of differential equations in three-dimensional phase spaces were used in the analysis. The calculations were performed by computer.

1. The equations of nonisothermal Couette flow of a non-Newtonian fluid with a power rheological law and the Reynolds relation for the effective viscosity of the fluid in a field of variable pressure have the form [15]

$$\lambda d^2T/dy^2 + \tau^2/\mu = 0; \tag{1.1}$$

$$\tau = \mu du/dy; \tag{1.2}$$

$$\mu = \mu_1 |du/dy|^{n-1}; \tag{1.3}$$

$$\mu_1 = \mu_0 e^{-\beta T}; \tag{1.4}$$

$$d\tau/dy = A, \tag{1.5}$$

where λ is the thermal conductivity, T is the temperature, y is the coordinate perpendicular to the surfaces of the plates, τ is the tangential stress, μ_1 is the effective viscosity, u is the velocity, and μ_0, A, β, and n are constant parameters.

After introducing the variables v = du/dy and w = dT/dy, Eqs. (1.1)-(1.5) can be reduced to an autonomous system of three first-order differential equations

$$\frac{dw}{dy} = -\frac{\mu_1}{\lambda} v|v|^n \operatorname{sign} v; \tag{1.6}$$

$$\frac{dv}{dy} = \frac{\beta}{n} vw + \frac{A}{\mu_1 n} \frac{v \operatorname{sign} v}{|v|^n}; \tag{1.7}$$

$$\frac{d\mu_1}{dy} = -\beta\mu_1 w. \tag{1.8}$$

An analysis of the dimensions of the variables in Eqs. (1.6) and (1.7) shows that the dimensions of velocity and length are related to those of the parameters in the following way:

$$\frac{[\beta][\mu_0]}{[\lambda]} = \frac{[y]^{n-1}}{[u]^{n+1}}, \quad \frac{[A]}{[\mu_0]} = \frac{[u]^n}{[y]^{n+1}}. \tag{1.9}$$

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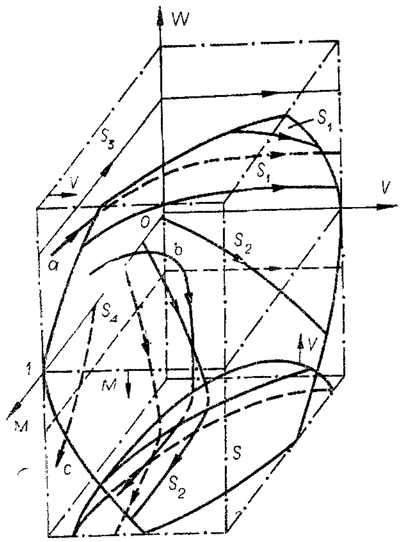

Fig. 1

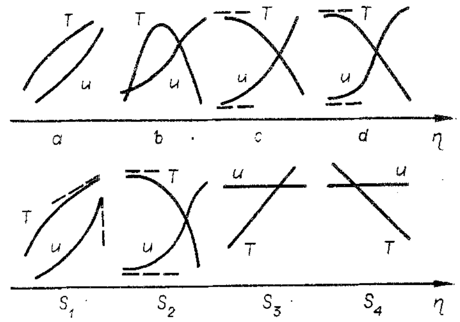

Fig. 2

Solving Eqs. (1.9) for the dimensions of velocity and length, we obtain

$$[u]=\frac{[\lambda]^{\frac{n+1}{3n+1}}[A]^{\frac{1-n}{3n+1}}}{[\beta]^{\frac{n+1}{3n+1}}[\mu_0]^{\frac{2}{3n+1}}},\quad [y]=\frac{[\lambda]^{\frac{n}{3n+1}}[\mu_0]^{\frac{1}{3n+1}}}{[\beta]^{\frac{n}{3n+1}}[A]^{\frac{n+1}{3n+1}}}.$$

Consequently, we can introduce the dimensionless variables

$$\eta=y\frac{A^{\frac{n+1}{3n+1}}}{\left(\frac{\lambda}{\beta}\right)^{n/(3n+1)}\mu_0^{1/(3n+1)}},\quad V=v\frac{\beta^{\frac{1}{3n+1}}\mu_0^{\frac{3}{3n+1}}}{\lambda^{1/(3n+1)}A^{2/(3n+1)}},$$

$$W=w\frac{\beta\lambda^{\frac{n}{3n+1}}\mu_0^{\frac{1}{3n+1}}}{A^{(n+1)/(3n+1)}\beta^{n/(3n+1)}},\quad M=\frac{\mu_1}{\mu_0}.$$

Rewriting Eqs. (1.6)-(1.8) in terms of these variables, we obtain

$$\frac{dW}{d\eta}=-MV|V|^n\operatorname{sign}V; \tag{1.10}$$

$$\frac{dV}{d\eta}=\frac{VW}{n}+\frac{V\operatorname{sign}V}{Mn|V|^n}; \tag{1.11}$$

$$\frac{dM}{d\eta}=-MW. \tag{1.12}$$

It follows from the form of system (1.10)-(1.12) that topologically different three-dimensional phase spaces correspond to $n < 1$ and $n \geqslant 1$. For any value of n the system (1.10)-(1.12) is invariant with respect to the symmetry transformation

$$\eta\to-\eta,\ V\to-V,\ W\to-W.$$

2. The phase space of flows of pseudoplastic liquids ($n < 1$) has an integral surface $V = 0$ which divides the phase space into two domains which are not connected by trajectories. This permits limiting the discussion to those parts of the phase space where $V \geqslant 0$, $M > 0$. The remaining solutions of system (1.10)-(1.12) for $n < 1$ can be obtained from the solutions of this part of phase space by a symmetry transformation.

The null surfaces of phase space are the coordinate $V = 0$ ($dW/d\eta = 0$, $dV/d\eta = 0$), $M = 0$ ($dM/d\eta = 0$, $dW/d\eta = 0$), $W = 0$ ($dM/d\eta = 0$), and the surface $W = -\operatorname{sign} V/M|V|^n$ ($dV/d\eta = 0$). The arrows on the null surfaces (Fig. 1) indicate the domains of positive values of the cor-

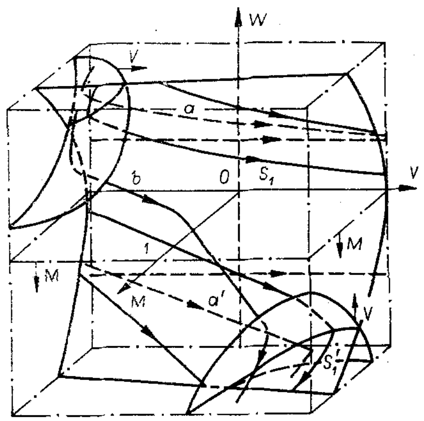

Fig. 3

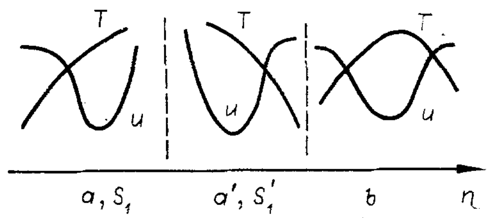

Fig. 4

responding derivatives. The equilibrium positions forming the straight line $V = 0$, $W = 0$ are unstable. The planes $V = 0$, $M = 0$ are integral surfaces formed by trajectories parallel respectively to the $W = 0$, $V = 0$ and $W = 0$, $M = 0$ axes. The phase space has a separating integral surface S spanning the axes $W = 0$, $M = 0$, and $V = 0$, $W = 0$, which together with the planes $V = 0$, $M = 0$ divides the phase space into three domains (Fig. 1). The integral curves corresponding to the trajectories of these domains are shown in Fig. 2. It should be noted that in the present case there is no fixed relation between the point of inflection on the velocity profile (Fig. 2, type b) and the maximum in the temperature profile which is characteristic for fluid flow without a gradient [5].

The graphs of Fig. 2, considered in a direction opposite the η axis, represent integral curves of the domain $V < 0$, $M > 0$ of phase space.

3. In the phase space of flows of Newtonian and dilatant fluids ($n \geqslant 1$) under the influence of a pressure gradient there is no line of equilibrium positions $V = 0$, $W = 0$, and the plane $V = 0$ is neither a null nor an integral surface (the remaining null surfaces have qualitatively the same form as for $n < 1$). In this case the trajectories of the half-space $M > 0$ cross over the domain $V < 0$ into the domain $V > 0$. The phase space has a separating surface S passing through the axis $W = 0$, $M = 0$. Together with the integral plane $M = 0$ it divides the phase space into three domains of qualitatively different trajectories (Fig. 3). The integral curves corresponding to these trajectories are shown in Fig. 4.

The extremal character of the velocity profile is a characteristic feature of the solutions of system (1.10)-(1.12) for $n \geqslant 1$. The fluid velocity in the core flow is lower than at the walls. This kind of velocity profile exists for both monotonic and nonmonotonic variations of the fluid temperature across the flow.

It should be noted that the transformation $\eta \to -\eta$, $W \to -W$, $V \to -V$, with respect to which system (1.10)-(1.12) is invariant, converts solutions of the type a, S_1, respectively, into solutions a', S_1', and solutions of type b into themselves.

4. As an example, Fig. 5 shows the results of solving system (1.10)-(1.12) simultaneously with the equations $V = d(u/[u])/d\eta$, $M = e^{-\beta T}$ for the Couette flow of a dilatant fluid ($n = 1, 2$) for various values of the temperature gradient and for the same values of the velocity gradient and the temperature at the stationary plate ($W_{01} = 1.5$, curve 1; $W_{02} = 2$, curve 2; $W_{03} = 2.5$, curve 3, $V_0 = -5$, $M_0 = 0.1$). The calculation confirms the conclusion of the qualitative study of the existence of type b solutions of Fig. 4 with two points of inflection and a minimum in the temperature profile. The maximum values of the temperature are reached in the domain of stationary fluid in the core flow. Figure 5 shows that the higher the velocity of the moving plate the higher the velocity of the fluid opposite the direction of motion of the plate in the zone of reverse current of the fluid where the effect of the pressure gradient on the motion of the fluid predominates. An increase in the temperature drop between the plates also leads to an increase in the temperature in the region where the fluid is slowing down, and these temperatures are higher than those on the plates. It should be noted that an increase in velocity of the moving plate for a given temperature of the stationary plate and steady flow involves a decrease in temperature of the moving plate. Solutions of the type a, S_1, Fig. 4 (and also a', S_1' by applying the trans-

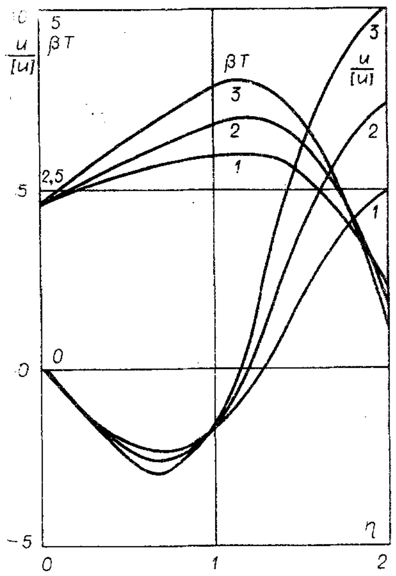

Fig. 5

formation $\eta \to -\eta$, $V \to -V$, $W \to -W$) can be obtained by displacing the moving plate along the η axis and making appropriate changes in its velocity and temperature.

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